ANOMALOUS SCALING EXPONENTS OF A WHITE-ADVECTED PASSIVE SCALAR

M. Chertkov and G. Falkovich
Department of Physics of Complex Systems, Weizmann Inst. of Science, Rehovot 76100, Israel

For Kraichnan's problem of passive scalar advection by a velocity field delta-correlated in time, the limit of large space dimensionality $d \gg 1$ is considered. Scaling exponents of the scalar field are analytically found to be $\zeta_{2n} = n\zeta_2 - 2(2-\zeta_2)n(n-1)/d$, while those of the dissipation field are $\mu_n = -2(2-\zeta_2)n(n-1)/d$ for orders $n \ll d$. The refined similarity hypothesis $\zeta_{2n} = n\zeta_2 + \mu_n$ is thus established by a straightforward calculation for the case considered.

It is likely that Kraichnan's model of white-advected passive scalar [1] will become a paradigm in theoretical studies of intermittency and anomalous scaling in turbulence [2–4]. This is because any simultaneous correlation function of a scalar satisfies a closed linear differential equation so that all common hypotheses about intermittency could, in principle, be verified by direct calculation. In an isotropic turbulence, the $2n$-point correlation function depends on $n(2n-1)$ distances, which makes direct solution of the respective partial differential equation quite difficult in a general case. The fourth-order correlation function has been calculated recently in two limiting cases: i) large space dimensionality $d \gg 1$ [3] and ii) almost smooth scalar field $2-\zeta_2 \ll 1$ [4]. By ζ_{2n} we designate the leading scaling exponent of the structure function: $S_{2n}(r_{12}) = \langle(\theta_1-\theta_2)^{2n}\rangle \propto r_{12}^{\zeta_{2n}} +$ subleading terms. If $\zeta_{2n} \neq n\zeta_2$ then the scaling is called anomalous. Anomalous scaling is the way intermittency manifests itself in developed turbulence: the degree of non-Gaussianity, that may be characterized by the dimensionless ratios S_{2n}/S_2^n, depend on scale. We shall see below that the anomalous dimensions $\Delta_{2n} = n\zeta_2 - \zeta_{2n}$ are positive which means that the smaller the scale of fluctuations the more non-Gaussian the statistics is.

In this Letter, we use the formalism developed in Ref. 3 to calculate high-order correlation functions assuming $1/d$ to be the smallest parameter in the problem. Following [3–5], we demonstrate how the anomalous part of the solution appears as a zero mode with the form independent of the pumping. Those zero modes exploit the interchange symmetry between the distances, the number of zero modes and anomalous exponents thus increases with the order of the correlation function.

The advection of a passive scalar field $\theta(t,\mathbf{r})$ by an incompressible turbulent flow is governed by the equation

$$(\partial_t + u^\alpha \nabla^\alpha - \kappa\triangle)\theta = \phi, \tag{1}$$

$\nabla^\alpha u^\alpha = 0$ being assumed. The external velocity $\mathbf{u}(t,\mathbf{r})$ and the source $\phi(t,\mathbf{r})$ are independent random functions of t and $\mathbf{r}$, both Gaussian and δ-correlated in time. Their spatial characteristics are different. The source is spatially correlated on a scale L i.e. the pair correlation function $\langle\phi(t_1,\mathbf{r}_1)\phi(t_2,\mathbf{r}_2)\rangle = \delta(t_1-t_2)\chi(r_{12})$ as a function of the argument $r_{12} \equiv |\mathbf{r}_1 - \mathbf{r}_2|$ decays on the scale L. The value $\chi(0) = P$ is the production rate of θ^2. The velocity field is multi-scale in space with a power spectrum. The pair correlation function $\langle u^\alpha(t_1,\mathbf{r}_1)u^\beta(t_2,\mathbf{r}_2)\rangle = \delta(t_1-t_2)[V_0\delta^{\alpha\beta} - \mathcal{K}^{\alpha\beta}(\mathbf{r}_{12})]$ is expressed via the so-called eddy diffusivity

$$\mathcal{K}^{\alpha\beta} = \frac{D}{r^\gamma}(r^2\delta^{\alpha\beta} - r^\alpha r^\beta) + \frac{D(d-1)}{2-\gamma}\delta^{\alpha\beta} r^{2-\gamma},$$

where $0 < \gamma < 2$ and isotropy is assumed.

Considering steady state and averaging (1) over the statistics of $\mathbf{u}$ and ϕ [6,7], one gets the closed balance equation for the simultaneous correlation function of the scalar $F_{1\ldots 2n} = F(\mathbf{r}_1,\ldots,\mathbf{r}_{2n}) = \langle\theta(\mathbf{r}_1),\ldots,\theta(\mathbf{r}_{2n})\rangle$:

$$-\hat{\mathcal{L}}F_{1\ldots 2n} = F_{1\cdots 2n-2}\chi_{2n-1,2n} + \text{permutations}. \tag{2}$$

The operator $\hat{\mathcal{L}} \equiv \sum_{i,j}\mathcal{K}^{\alpha\beta}(\mathbf{r}_{ij})\nabla_i^\alpha\nabla_j^\beta/2 + \kappa\sum\triangle_i$ describes both turbulent and molecular diffusion, it may be rewritten in terms of relative distances r_{ij} [3]:

$$\hat{\mathcal{L}} = \frac{D(d-1)}{2-\gamma}\sum_{i>j} r_{ij}^{1-d}\partial_{r_{ij}}\left(r_{ij}^{2-\gamma} + r_d^{2-\gamma}\right)r_{ij}^{d-1}\partial_{r_{ij}}$$
$$-\frac{D(d-1)}{2(2-\gamma)}\sum(r_{in}^2 - r_{ij}^2 - r_{jn}^2)\frac{r_{ij}^{1-\gamma}}{r_{jn}}\frac{\partial^2}{\partial r_{ij}\partial r_{jn}}$$
$$-\frac{D}{4}\sum\frac{1}{r_{ij}^\gamma r_{im} r_{jn}}\left(\frac{d+1-\gamma}{2-\gamma}r_{ij}^2(r_{in}^2 + r_{jm}^2 - r_{ij}^2 - r_{mn}^2)\right.$$
$$\left.+\frac{1}{2}(r_{ij}^2 + r_{im}^2 - r_{jm}^2)(r_{ij}^2 + r_{jn}^2 - r_{in}^2)\right)\frac{\partial^2}{\partial r_{im}\partial r_{jn}}$$
$$+\kappa\sum\frac{r_{ij}^2 + r_{im}^2 - r_{mj}^2}{2r_{ij}r_{im}}\frac{\partial^2}{\partial r_{ij}\partial r_{im}}. \tag{3}$$

Here, the summation is performed over $n(2n-1)$ independent distances (for $d > 2n-2$) with subscripts satisfying the conditions $i \neq j$ and $m \neq i,j$, $n \neq i,j$; the diffusion scale $r_d^{2-\gamma} = 2\kappa(2-\gamma)/(D(d-1)$ has been introduced. We consider the convective interval $L \gg r_{ij} \gg r_d$ where the operator $\hat{\mathcal{L}}$ is scale invariant. Taking $n=1$, it is easy to find the pair correlation functions of the scalar [1,3]:

$$F_{12}(r) = P\frac{2-\gamma}{\gamma(d-1)D}\left(\frac{L^\gamma}{d-\gamma} - \frac{r^\gamma}{d}\right). \tag{4}$$

We see that $\zeta_2 = \gamma$. The scaling exponent of $\hat{\mathcal{L}}$ is $-\gamma$; the solution of (2) may thus be presented in the form

$F=F_{\rm forc}+\mathcal{Z}$, where we separated the so-called "forced" part of the solution (with the scaling exponent $\zeta_{2n-1}+\gamma$ prescribed by the rhs) from the zero mode $\mathcal{Z}$ that may have a different scaling. It has been recognized independently by the authors of [3–5] that they are the zero modes of the operator $\hat{\mathcal{L}}$ that are responsible for the anomalous scaling. We shall demonstrate below that the factor $(L/r)^{\Delta_{2n}}$ appears in the zero mode while the factor $(L/r)^{\Delta_{2n-2}}$ appears in the forced term, Δ_{2n} being the anomalous dimension. The zero mode turns out to be dominant. It has been demonstrated in [3] that r_d does not appear in the leading terms of F as long as at least some of the distances r_{ij} are in the convective interval. Assuming that to be the case, we omit the diffusive parts of $\hat{\mathcal{L}}$. We shall account for them later while considering the correlation functions of the scalar derivatives.

Let us consider now the case of large space dimensionality where the anomalous dimensions can be calculated analytically. It is seen from (4) that the level of scalar fluctuations necessary to provide for a given flux P decreases as d increases. It may be shown that $F_{1\ldots 2n}\propto P^n/[d(d-1)]^n$. Considering large d and assuming that the flux is determined by the pumping (and it is thus d-independent) we shall develop the perturbation theory for the quantities $d^{2n}F_{1\ldots 2n}$ which have finite limits at $d\to\infty$. We shall show below that despite the small level of fluctuations at large d, the statistics of the scalar is substantially non-Gaussian at small scales. However, since the anomalous dimensions are small, there exists a wide interval of scales where the correlation functions are close to their Gaussian values and can be calculated by perturbation theory to obtain Δ_{2n} at the leading order in $1/d$. As $d\to\infty$, the main part of $\hat{\mathcal{L}}$ is the operator of the first order $\hat{\mathcal{L}}_0=(d^2D/(2-\gamma))\sum r_{ij}^{1-\gamma}\partial_{r_{ij}}$. Since $\hat{\mathcal{L}}$ is of the second order, one may wonder if the $1/d$ perturbation theory is regular. It has been demonstrated in [3] that it is so by considering the perturbation theory that starts from the bare operator $\hat{\mathcal{L}}_0'\propto\sum r_{ij}^{1-d}\partial_{r_{ij}}r_{ij}^{1+d-\gamma}\partial_{r_{ij}}$; this alternative theory gives the same answer for the anomalous exponent. The difference between the perturbation theories with $\hat{\mathcal{L}}_0$ and $\hat{\mathcal{L}}_0'$ is in the representation for the forced term (the second approach corresponds to some resummation). Since we are interested here in the anomalous scaling which is given by the zero modes that are determined by their behavior at $r=0$, there is no qualitative difference between $\hat{\mathcal{L}}_0$ and $\hat{\mathcal{L}}_0'$ and the perturbation theory for the anomalous exponent is regular.

The zeroth term in the perturbation series for $F_{1\ldots 2n}$ is given by a Gaussian reducible expression. Our aim is to iterate it once by applying the operator $\hat{\mathcal{L}}_0^{-1}(\hat{\mathcal{L}}-\hat{\mathcal{L}}_0)$. The parameter of the expansion is $n/\gamma d$, assumed to be small. In the first correction to be thus found, the logarithmic terms $\ln(L/r_{ij})$ are of interest because they appear at expanding the anomalous scaling factors $(L/r_{ij})^{\Delta_{2n}}$ over Δ_{2n} — see [3] for the details. As we shall see, there are different zero modes with different anomalous dimensions $\Delta_{2n,i}$ for any given n. Of course, only the largest $\Delta_{2n,i}$ contributes in the limit $L/r\to\infty$. However, in the region $\Delta_{2n,i}\ln(L/r_{ij})\ll 1$, where we carry out our calculations, all Δ's contribute logarithmic terms. We thus have a degenerate perturbation theory (all zero modes $\mathcal{Z}_{2n,i}$ have the same exponent $n\gamma$ in the zeroth approximation) and should therefore apply the operator $\hat{\mathcal{L}}_0^{-1}(\hat{\mathcal{L}}-\hat{\mathcal{L}}_0)$ on a vector of zero modes of $\hat{\mathcal{L}}_0$ and then single out the terms having logarithms. It is clear that logarithms may appear only multiplied by a zero mode of $\hat{\mathcal{L}}_0$. We thus obtain the matrix of the operator $\hat{\mathcal{L}}_0^{-1}(\hat{\mathcal{L}}-\hat{\mathcal{L}}_0)$ in the representation of $\mathcal{Z}_{2n,i}$. The eigenvalues of that matrix are the anomalous exponents $\Delta_{2n,i}$ at the leading order in $1/d$.

Let us describe how the matrix is generated. The most convenient classification of the zero modes is as follows: $\mathcal{Z}_{2n,i}$ is the polynomial in $x=r^\gamma$ of order n which may be separated into a symmetrical sum of polynomials, each involving distances between i points. For example, there are two zero modes for the fourth-order correlator: $\mathcal{Z}_{4,4}=\sum(x_{ij}-x_{kl})^2$ and $\mathcal{Z}_{4,3}=\sum(x_{ij}-x_{jk})^2$ [3]. The first-order logarithmic correction is calculated by the rule

$$-\hat{\mathcal{L}}_0^{-1}(\hat{\mathcal{L}}-\hat{\mathcal{L}}_0)x_{ij}x_{kl}=\frac{2-\gamma}{2d}\ln[r]\begin{cases}(x_{ij}-x_{il})^2+\gamma(x_{ij}-x_{jl})(x_{il}-x_{jl}), & i=k,\quad j\neq l;\\ 2(x_{ik}-x_{il}-x_{jk}+x_{jl})^2, & i\neq j\neq k\neq l,\end{cases}\tag{5}$$

which gives the matrix [3]

$$\begin{pmatrix}\Delta_{4,4} & \ldots\\ 0 & \Delta_{4,3}\end{pmatrix}.\tag{6}$$

Here, $\Delta_{4,4}=4(2-\gamma)/d$ and $\Delta_{4,3}=-(4-\gamma^2)/2d$ are the eigenvalues. Before describing the structure of the matrix at higher n, let us note that the order n and the number of points i are not enough to specify the zero mode for $n\geq 3$ and $4\leq i\leq 2n-2$ due to the possibility of different topological configurations (we enumerate them by j). For example, at $n=3, i=4$, there are two different zero modes: one involves distances r_{kl},r_{lm},r_{mn} and another r_{kl},r_{ln},r_{lm}. The total number of zero modes grows with n faster than factorially due both to the growth of the number of possible functional forms and the appearance of new topologically different configurations. What is important to know is that the operator acting on the zero mode with a given i produces only modes with $i'\leq i$. The analysis of eigenvalues is thus reduced to the consideration of the blocks with a given i. The first mode $\mathcal{Z}_{2n,2n}$ contains the monomial $x_{1,2}\cdots x_{n-1,n}$ that cannot be obtained by (5) from other modes, so that the first element

is $n(n-1)\Delta_{4,4}/2$ and the remaining elements of the first column are zero. Considering $\mathcal{Z}_{2n,2n-1}$, one realizes that the second diagonal element is $(n-1)(n-2)\Delta_{4,4}/2+\Delta_{4,3}$ and all the lower elements in the second column are zero. Then the 3×3 block follows which corresponds to $\mathcal{Z}_{2n,2n-2,j}$:

$$\frac{\Delta_{4,4}}{2}\begin{pmatrix}(n-1)(n-4)+4q & 0 & -2\\ 2(3-n) & n^2-3n-1-6q & 1+4q\\ 0 & 9+12q & (n-2)(n-5)\end{pmatrix}, \quad (7)$$

with $q=\Delta_{4,3}/\Delta_{4,4}$, all the elements below the block are zero. The next block is for $i=2n-3$, it is 7×7 for $n\geq 6$, the sizes of the blocks grow as one approaches the center of the matrix, then they decrease and eventually we come to the 3×3 block due to $\mathcal{Z}_{2n,2,j}$ and a single value at the lower right corner. By virtue of (5), all entries of the matrix at arbitrary n may be expressed via those $\Delta_{4,4}$ and $\Delta_{4,3}$ and combinatorial factors. Note that the classification of all eigenvalues and eigenvectors is very important because it carries information about the algebraic structure and underlying symmetry that governs our set of correlation functions. We postpone the general classification until further detailed publications. Fortunately enough, the mode that gives the largest anomalous dimension and the structure functions of the dissipation field is separated so that it can be found without finding the whole set of $\Delta_{2n,i}$. We notice that since $\Delta_{4,4}>\Delta_{4,3}$ and the largest combinatorial factor in front of $\Delta_{4,4}$ appears in the first element then it is plausible to assume that $\mathcal{Z}_{2n,2n}$ gives the largest eigenvalue $\Delta_{2n,2n}=n(n-1)\Delta_{4,4}/2$. One can directly check that $\Delta_{2n,2n-1}$ and all eigenvalues $\Delta_{2n,2n-2}$ of (7) are less than $\Delta_{2n,2n}$ for any n. For an arbitrary block, the validity of the assumption may be established asymptotically for $n\gg 1$ (yet, of course, $n\ll\gamma d$) when all eigenvalues are $n^2\Delta_{4,4}/2+O(n)$. For $n=2,3,4$, we found all the eigenvalues using Mathematica; the largest is always $\Delta_{2n,2n}$. We thus conclude that for $\Delta_{2n,2n}\ln(L/r)\gg 1$, the main contribution to the correlation function is given by the respective zero mode with the scaling exponent $\zeta_{2n}=n\zeta_2-2(2-\zeta_2)n(n-1)/d$. In particular,

$$\langle(\theta_1-\theta_2)^{2n}\rangle\sim r^{n\gamma}(L/r)^{2n(n-1)(2-\gamma)/d}. \quad (8)$$

It agrees with [4] where ζ_4 has been calculated. Note that both (8) and the results of [3,4] differ from ζ_{2n} suggested in [2]. In our opinion, that means that the closure implemented in [2] is not exact at the limits considered ($n\ll\gamma d$ and $2-\gamma\ll 1$).

To find the correlation functions of the scalar derivatives, one should consider some distances r_{ij} as going to zero. While some distance passes the diffusion scale the dependence on that distance changes. To describe that, we include the diffusion operator into $\hat{\mathcal{L}}$. As a result, the diffusion scale r_d appears in the correlation functions. The form of r_d dependence could be readily established for an arbitrary n,d,γ by using a straightforward perturbation expansion in the ratio between small and large distances (see Sect. III of [3]). The overall scaling of $F_{1\cdots 2n}$ at all the distances inside the convective interval is assumed now to be known ($\Delta_{2n}=\Delta_{2n,2n}$ for $d\gg 1$)

$$F_{1\cdots 2n}\approx C_{2n}R^{n\gamma-\Delta_{2n}}L^{\Delta_{2n}}, \quad L\gg r_{ij}\sim R\gg r_d\ . \quad (9)$$

Now, let us consider one distance, say r_{12}, to be much smaller than the others: $\rho=r_{12}\ll r_{ij}\simeq R$. At zero order in ρ, $F_{1,1,3\cdots 2n}\approx G(R)\sim R^{n\gamma-\Delta_{2n}}L^{\Delta_{2n}}$ [3]. The leading isotropic correction satisfies the equation

$$\hat{\mathcal{L}}_{\boldsymbol{\rho}}\delta F_{2n}(\mathbf{R},\boldsymbol{\rho})=\hat{\mathcal{L}}_R G(R)-\Phi(R) \quad (10)$$

where $\Phi(R)\sim R^{(n-1)\gamma-\Delta_{2n-2}}L^{\Delta_{2n-2}}$ is the major (R-dependent) term of the rhs of (2), $\hat{\mathcal{L}}_R$ is the major term of the operator $\hat{\mathcal{L}}$, and $\hat{\mathcal{L}}_{\boldsymbol{\rho}}$ is the perturbation operator $\hat{\mathcal{L}}_{\boldsymbol{\rho}}\equiv\mathcal{K}^{\alpha\beta}(\boldsymbol{\rho})\nabla^\alpha_\rho\nabla^\beta_\rho+2\kappa\triangle_\rho,$. The solution has the form

$$\delta F_{2n}(\mathbf{R},\boldsymbol{\rho})\sim R^{(n-1)\gamma-\Delta_{2n}}L^{\Delta_{2n}}\int_0^\rho\frac{rdr}{2r_d^{2-\gamma}+r^{2-\gamma}}, \quad (11)$$

In deriving (11), it has been implied that $\Delta_{2n}>\Delta_{2n-2}$. At $\rho\ll r_d$, the isotropic correction is analytic in ρ: $\delta F_{2n}(\mathbf{R},\boldsymbol{\rho})\sim R^{(n-1)\gamma-\Delta_{2n}}L^{\Delta_{2n}}r_d^{\gamma-2}\rho^2$. Now we are ready to differentiate it with respect to ρ, in particular, to calculate the correlation functions that involve the dissipation field $\epsilon=\kappa[\nabla\theta]^2$:

$$\langle\langle\epsilon_1\theta_3\cdots\theta_{2n}\rangle\rangle\sim R^{(n-1)\gamma}\big(L/R\big)^{\Delta_{2n}}\ . \quad (12)$$

Since $\epsilon\propto\kappa\propto r_d^{2-\gamma}$ then r_d disappears. The diffusion scale appears only at the anisotropic terms proportional to angular harmonics with respect to the angle between $\mathbf{R}$ and $\boldsymbol{\rho}$. This is because of the zero modes of the operator $\hat{\mathcal{L}}_{\boldsymbol{\rho}}$ associated with the angular harmonics that are expressed via the Jacobi polynomials $P_{2k}^{(\nu,\nu)}$ [3,8]:

$$\mathcal{Z}_{2k}(\boldsymbol{\rho},\mathbf{R})=P_{2k}^{(\nu,\nu)}\left[\frac{\mathbf{n}\boldsymbol{\rho}}{\rho}\right]\times\begin{cases}\rho^{\delta_k}, & \rho\gg r_d,\\ \rho^{2k}r_d^{\delta_k-2k}, & \rho\ll r_d;\end{cases},$$

$$\delta_k=\frac{1}{2}\left(\gamma-d+\sqrt{(d-\gamma)^2+\frac{8k(d+1-\gamma)(2k+d-2)}{d-1}}\right),$$

where $\nu=(d-3)/2$, $\mathbf{n}=\mathbf{R}/R$. The $2k$-th angular correction to $F_{1,1,3,\cdots,2n}$ is thus

$$\delta^{(k)}F_{2n}(\boldsymbol{\rho},\mathbf{R})\sim\mathcal{Z}_{2k}(\boldsymbol{\rho},\mathbf{R})\big(L/R\big)^{\Delta_{2n}}R^{n\gamma-\delta_k}, \quad (13)$$

where the leading scaling behavior of the ρ-independent prefactor is restored to give a correct dimensionality. The overall (L-dependent) scaling is assumed to be known. If $n\gamma>\delta_k$ and ρ is in the diffusive interval, then the estimate (13) is true only at R being small enough: $(R/L)^{\Delta_{2n}}(r_d/R)^{n\gamma-\delta_k}\ll 1$ — see [3] for the details.

Differentiating $\delta^{(2k)}F$ with respect to $\boldsymbol{\rho}$ one can establish the scaling of the correlation functions that involve high-order traceless tensors of derivatives $\xi^{(k)} = \kappa^{(2k-\gamma)/(2-\gamma)} \sum_{l=0}^{k} a_l \left[(\mathbf{n}\nabla)^{k-l}\nabla^{\alpha_1}\cdots\nabla^{\alpha_l}\theta\right]^2$, a_l is a respective coefficient of $P_{2k}^{(\nu,\nu)}(x)$ expansion in the series in x. One can also generalize (12) in another way, considering extra fusion of another one or more pairs of points. The results show that the fusion procedures for an arbitrary number of different pairs of points commute with each other. In other words, using the language of the so-called operator algebra [9–11] the validity of which for turbulence was argued in [12,13], we can introduce r_d-related dimensionality of the fields: the scalar field θ and dissipation field ϵ have dimensionality 0, while $\xi^{(k)}$ has dimensionality $\delta_k - \gamma$. Thus, to find the ultraviolet (r_d-related) dimensionality of a composite multiplicative field one should sum the dimensionalities of the multipliers:

$$\left\langle \theta_1\cdots\theta_l\epsilon_{l+1}\cdots\epsilon_m \prod_k [\xi^{(k)}]^{c_k}\right\rangle \propto r_d{}^{\sum c_k(\gamma-\delta_k)} .$$

The above fusion rules describe the scaling of the fluctuations of the enumerated primary fields only. Particularly, the rules give the scaling of correlation functions with ϵ at different points but not with the higher powers of the dissipation field, ϵ^n. The difference stems from the fact that to calculate the correlation functions containing ϵ^n we should fuse a group of $2n$ (rather than two) points in an initial θ correlator. To calculate a correction that produces a nonzero contribution into the desirable correlator of ϵ^n, one should compare the leading forced correction with a zero mode of the $2n$-point operator, but not with that of two-point one $\hat{\mathcal{L}}_{\boldsymbol{\rho}}$. Such zero modes do have anomalous scaling, as we have learned above, they produce the major contribution to correlation functions of ϵ^n. The ultraviolet anomalous scaling of such objects is thus related to the infrared anomalous scaling of the passive scalar itself. For example, by a direct application of the above procedure (10-12), we get

$$\langle\epsilon_1^n\epsilon_2^m\rangle \sim (L/r_{12})^{\Delta_{2n+2m}-\Delta_{2n}-\Delta_{2m}}(L/r_d)^{\Delta_{2n}+\Delta_{2m}} . \quad (14)$$

The dissipation field is thus highly intermittent, the single-point means $\langle\epsilon^n\rangle \sim \langle\epsilon\rangle^n(L/r_d)^{\Delta_{2n}}$ grow unlimited when diffusivity decreases. Statistics at the convective interval is better related to local average ϵ_r over the ball with the radius r. Since spatial integration and time average commute then our knowledge of $\langle\epsilon_1\cdots\epsilon_n\rangle$ with all distances in the convective interval allows one to obtain by spatial integration (which converges if $\Delta_4 < d$):

$$\langle(\epsilon_r)^n\rangle \sim \langle\epsilon\rangle^n(r/L)^{\mu_n},\ \mu_n = \zeta_{2n} - n\zeta_2 = -\Delta_{2n} . \quad (15)$$

That relation presents a version of the refined similarity hypothesis [15–17] valid in our case. Indeed, scalar field and dissipation field are related: $\epsilon_r \simeq (\delta\theta_r)^2/\tau_r$ where $\tau_r \simeq Dr^\gamma$ is a transfer time. For $n=2$, (15) has been established in [2,3].

The first ($n \ll \gamma d$) moments of the locally averaged dissipation have $\mu_n = -n(n-1)\Delta_{4,4}/2$ so they are described by *log*-normal statistics [14]:

$$\mathcal{P}(\epsilon_r) \sim \exp\left(-\frac{(\ln[\epsilon_r/\langle\epsilon\rangle] - \Delta_{4,4}\ln[L/r]/2)^2}{2\Delta_{4,4}\ln[L/r]}\right) .$$

To conclude, the law (8) qualitatively corresponds to the observed behavior of ζ_n [18–20]. Quantitatively, we cannot use (8) for $d=3, \gamma=2/3$ (which would give an overestimation of the anomalous exponents) because the validity condition of our theory $n \ll \gamma d$ will be violated already at $n=2$. Note that quadratic dependence of ζ_n and μ_n on n (and lognormality) is violated when $n \simeq \gamma d$, while the similarity relation (15) is true for any n, γ, d.

This work is a part of the extensive program on studying anomalous scaling in turbulence undertaken together with E. Balkovsky, I. Kolokolov and V. Lebedev. We are grateful to them for numerous discussions. We are indebted to G. Eyink, U. Frisch, R. Kraichnan and the anonymous referee for useful remarks. The work was supported by the Clore Foundation (M.C.) and by the Rashi Foundation (G.F.).

[1] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).

[2] R. Kraichnan, V. Yakhot and S. Chen, Phys. Rev. Lett. **75**, 240 (1995).

[3] M. Chertkov, G. Falkovich, I. Kolokolov, V. Lebedev, Phys. Rev. E **52**, 4294 (1995).

[4] K. Gawędzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995)

[5] B. Shraiman and E. Siggia, C.R. Acad. Sci. Paris, **321**, 279 (1995).

[6] R. Kraichnan, private communication.

[7] B. Shraiman and E. Siggia, Phys. Rev. E **49**, 2912 (1994).

[8] A. Fairhall, O. Gat, V. Lvov and I. Procaccia, Phys. Rev. E (in press).

[9] A. M. Polyakov, Sov. Phys. JETP **30**, 151 (1970).

[10] L. P. Kadanoff, Phys. Rev. Lett. **23**, 1430 (1969).

[11] K. G. Wilson, Phys. Rev. **179**, 1499 (1969).

[12] G. Eyink, Phys. Lett. **A 72**, 355 (1993).

[13] V. Lebedev and V. L'vov, JETP Letters **59**, 577 (1994).

[14] A. N. Kolmogorov, Dokl. AN SSSR **31**, 99 (1941).

[15] A. N. Kolmogorov, J. Fluid. Mech. **12**, 82 (1962).

[16] A. M. Obukhov, J. Fluid. Mech. **12**, 77 (1962).

[17] G. Stolovitsky, P. Kailasnath and K.R. Sreenivasan, J. Fluid Mech. **297**, 275 (1995).

[18] R. Antonia, E. Hopfinger, Y. Gagne, and F. Anselmet, Phys. Rev. A **30**, 2704 (1984).

[19] C. Meneveau and K. R. Sreenivasan, Phys. Rev. A **41**, 2246 (1990).

[20] K. R. Sreenivasan, Proc. R. Soc. Lond. A **434**, 165 (1991).